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Schwinger Algebra for Quaternionic Quantum Mechanics

L.P. Horwitz[†]

School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences
 Tel Aviv University, Ramat Aviv, 69978 Israel

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Abstract: It is shown that the measurement algebra of Schwinger, a characterization of the properties of Pauli measurements of the first and second kinds, forming the foundation of his formulation of quantum mechanics over the complex field, has a quaternionic generalization. In this quaternionic measurement algebra some of the notions of quaternionic quantum mechanics are clarified. The conditions imposed on the form of the corresponding quantum field theory are studied, and the quantum fields are constructed. It is shown that the resulting quantum fields coincide with the fermion or boson annihilation-creation operators obtained by Razon and Horwitz in the limit in which the number of particles in physical states $N \rightarrow \infty$.

[†] Also at Department of Physics, Bar Ilan University, Ramat Gan Israel.

1. Introduction

Schwinger¹ has given an interpretation of the structure of the quantum theory, which is both deep and pedagogically useful, in terms of the algebraic properties of filters. These filters are considered to represent the selection of subensembles from a “beam” of independent, but identical, quantum systems, corresponding to the quantum ensemble containing the properties of a single system. The algebra of these filters is represented in a vector space that is the Hilbert space of the quantum theory. It is assumed that the coefficients which arise when a sequence of measurements of non-compatible observables is made are elements of the complex number field \mathbf{C} . In this work, I generalize this structure to admit elements of the quaternionic algebra \mathbf{H} , where, for $q \in \mathbf{H}$, the real algebra of quaternions (with involutory automorphism q^*),

$$q = \sum_0^3 \lambda_i e_i \quad (\lambda \text{ real});$$

$$e_i e_j = e_k (ijk \text{ cyclic}), \quad e_i^* = -e_i, \quad i \neq 0,$$

$$e_i e_j = -e_j e_i \quad (i \neq j \neq 0),$$

$$e_0 = 1, \text{ and } e_i^2 = -1,$$

and show that the resulting vector space is the Hilbert module corresponding to the quaternionic quantum theory^{2,3,4}.

Schwinger¹ proceeds to construct quantum field theory by considering each system as a composite of a set of identical subsystems. The corresponding set of measurement filters are then constructed as a (direct) sum of filters sensitive to each of the subsystems. Since the subsystem filters commute, the algebra may be closed as a commutator algebra, and the factorized solutions for the representations of these operators result in the definition of the quantum fields. In the case of quantum fields on the usual complex Hilbert space, the numerical coefficients corresponding to the transformation functions (transition amplitudes) factor out of the non-linear algebraic expressions, but in the non-commutative case that I consider in this work, the requirement of Bose-Einstein or Fermi-Dirac symmetry imposes a structure on the numerical coefficients according to which they must act distributively, as derivations. With this structure, I find that the quantum fields defined by Schwinger’s method correspond to the annihilation-creation operators of the Fock space constructed by Horwitz and Razon⁵, in the limit in which the physical states of the theory contain occupation number $N \rightarrow \infty$. It is only in this case that the commutation (anticommutation) relations are not deformed⁶, so that bilinears in these operators may generate closed Lie algebras.

In the following, I review Schwinger’s construction briefly for the complex (Abelian) case, and in Section 2, construct the corresponding quaternionic measurement algebra and its representation in terms of the quaternionic Hilbert module. In Section 3, the measurement algebra for the non-commutative quaternionic many-body system of identical particles is constructed, and in Section 4, I construct its representation in terms of quantum fields. Some conclusions and discussion are given in Section 5.

Schwinger considers a system defined by a set of observables A, B, C, \dots , with possible real values a', b', c', \dots . Let us assume the existence of filters¹, symbolized by $\{M(a')\}$,

and a *beam* of systems which have these properties; the filter $M(a')$ selects all systems with the property $A = a'$ from the beam and passes them with this observable having the exact value a' immediately after the measurement. This operation corresponds to a Pauli measurement of the first kind (the spectra $\{a'\}$ may be discrete or continuous; discussion of the latter case is technically somewhat more complicated, and we do not treat it explicitly here). It then follows that

$$M(a'')M(a') = \delta(a', a'')M(a'), \quad (1.1)$$

provided that the filters act exclusively on the possible values $\{a'\}$. In addition to this orthogonality property, the interpretation of addition as *or* is manifest in the completeness statement,

$$\sum_{a'} M(a') = I, \quad (1.2),$$

where I is the measurement which selects and transmits every system in the beam, without alteration. To realize this construction, one may think of storing each outcome of the action of $M(a')$ in a storage ring; repeating the process for each a' results in a stored beam identical to the original one.

Schwinger defines the Pauli measurement of the second kind, symbolized by $M(a', b')$; it corresponds to a filter which selects systems with value b' of the property B , and passes them through the filter with property A having definitely the value a' (e.g., systematic change of polarization).

If the properties A , B are not compatible, a measurement of the first kind selecting systems with value b' of property B leaves the set of systems with a distribution of possible values a' of A (eventually to be determined by the physical nature of the system). Following the B measurement, an A measurement leaves the system with a definite value a' of A . This operation is represented by

$$M(a')M(b') = \langle a'|b' \rangle M(a', b'), \quad (1.3)$$

since the selected systems have definite value b' of B , and are left with a definite value a' of A ; the uncertainty in finding a' in a system with the value b' of B reflects the microscopic indeterminacy of the quantum theory, and is represented by the coefficient $\langle a'|b' \rangle$. The closure of the algebra in the form

$$M(a', b')M(c', d') = \langle b'|c' \rangle M(a', d') \quad (1.4)$$

clearly depends on the assumption of Abelian $\{\langle b'|c' \rangle\}$. Using (1.2),

$$\begin{aligned} M(a')\delta(a', a'') &= \sum_{b'} M(a')M(b')M(a'') \\ &= \sum_{b'} \langle a'|b' \rangle \langle b'|a'' \rangle M(a', a''), \end{aligned} \quad (1.5)$$

where we have used $M(a', a') \equiv M(a')$. We therefore have the completeness relation

$$\delta(a', a'') = \sum_{b'} \langle a'|b' \rangle \langle b'|a'' \rangle. \quad (1.6)$$

Schwinger¹ defines a linear map from the set $M(a', b')$ to the number system of the coefficients $\langle a'|b' \rangle$,

$$\text{tr} M(a', b') = \langle b'|a' \rangle; \quad (1.7)$$

with the definition of an adjoint action (reversal of the second kind filtering process),

$$M(a', b')^\dagger = M(b', a'), \quad (1.8)$$

so that

$$\begin{aligned} \text{tr} M(b', a') &= \text{tr}(M(a', b')^\dagger) = (\text{tr} M(a', b'))^\dagger = \langle b'|a' \rangle^\dagger \\ &= \langle a'|b' \rangle; \end{aligned} \quad (1.9)$$

identifying this action on the number system with its involutory automorphism, (1.6), for $a' = a''$, becomes, if we assume that we are dealing with a normed algebra,

$$1 = \sum_{b'} |\langle b'|a' \rangle|^2. \quad (1.10)$$

It is then assumed that the $\{\langle b'|a' \rangle\}$ are in \mathbf{C} .

Schwinger then asks for a representation of this algebra in terms of a vector space. The factorized definition

$$M(a', b') = |a' \rangle \langle b'|, \quad (1.11)$$

with the rule $\langle b'| \cdot |c' \rangle = \langle b'|c' \rangle$, constituting a scalar product in this vector space, satisfies all the structural requirements imposed by the algebra. The completion of this vector space is the usual quantum mechanical Hilbert space over the complex field.

We do not wish to reproduce here the large set of consequences of the structure outlined above, but only to emphasize the basic properties of the algebra. As we shall see, the assumption that the $\{\langle a'|b' \rangle\} \in \mathbf{C}$ is not necessary; the quaternion skew-field \mathbf{H} provides an equally workable choice, and includes a higher level of intrinsic structure^{2,3,4}.

We now describe briefly the structure of the many-body theory, in the context of the complex number field. To deal with the many-body theory, Schwinger considers the situation in which each system (element of the beam) consists of a set of identical subsystems carrying the same set of properties A, B, C, \dots . Filters for such a beam can be constructed in a symmetrical way as

$$\mathbf{M}(a', b') = \sum_{i=1}^N M(a'_i, b'_i), \quad (1.12)$$

where each $M(a'_i, b'_i)$ is a filter of the second kind acting on the subsystem labelled by i . The entire system is passed if the requirements on any subsystem (that B of i have the value b' ; it is then transformed to a' of A) are met. The beam resulting from the sum

contains (relatively) as many systems as there were in the incoming beam which have a subsystem with the value b' of B (in correspondence with the action of an annihilation operator on a state in the Fock space); in the final state, these subsystems have value a' of A .

The action of $M(a'_i, b'_i)$ and $M(c'_j, d'_j)$, for $i \neq j$, commute, since the subsystems are considered to be dynamically independent. The algebra of the many-body measurement symbols (1.12) then closes as a *commutator* algebra in the form

$$\begin{aligned} \mathbf{M}(a', b')\mathbf{M}(c', d') - \mathbf{M}(c', d')\mathbf{M}(a', b') \\ = \langle b'|c' \rangle \mathbf{M}(a', d') - \langle d'|a' \rangle \mathbf{M}(c', b'), \end{aligned} \quad (1.13)$$

where the isomorphic structure of all the subsystems implies that $\langle a'_i|b'_i \rangle \equiv \langle a'|b' \rangle$ are all equal. The factorization

$$\mathbf{M}(a', b') = \psi(a')^\dagger \psi(b'), \quad (1.14)$$

where

$$[\psi(a'), \psi(b')^\dagger]_{\mp} = \langle a'|b' \rangle \quad (1.15)$$

provides an effective solution for the algebra (1.13). With this, Schwinger is able to identify the quantum fields.

In the succeeding sections, I discuss a generalization of this structure to the quaternionic skew-field.

This paper is dedicated to the memory of my former friend and colleague, Larry Biedenharn, with whom there were many years of collaboration in the study of hypercomplex quantum theories, among other subjects; it also intended as a tribute to his courageous wife, Sarah, who helped to make this collaboration a warm friendship.

2. Measurement algebra of the quaternionic quantum theory

To construct an analog of the Schwinger algebra which results in the Hilbert module of quaternionic quantum mechanics, we assume that there is a non-trivial (non-Abelian) degree of freedom² associated with the algebraic orientation of the number field associated with the final state of a system emerging from the filters corresponding to ideal measurements. Such a filter is empowered with the possibility of introducing an algebraic automorphism on the number system. I represent this as $M(a'; q; b')$, a generalized Pauli measurement of the second kind, where, in the quaternion case in which we are interested, $q \in \mathbf{H}$. It arises in the action

$$M(b')M(a') = M(b'; \langle b'|a' \rangle; a'), \quad (2.1)$$

where the number $\langle b'|a' \rangle$, a measure of the likelihood of finding b' among a set of systems with A having value a' , contains as well an essential non-Abelian phase. We assume, moreover, the linearity properties

$$\begin{aligned} \sum_i M(a'; q_i; b') &= M(a'; \sum_i q_i; b') \\ M(a'; \lambda q; b') &= \lambda M(a'; q; b'), \end{aligned} \quad (2.2)$$

for λ real.

The filter $M(b'; q; a')$ selects a system for which the observable A has the value a' and passes this system with observable B having, immediately after the measurement, value b' generating, at the same time, an automorphism induced by q on the structure of the number system (in this case, quaternionic) to be used for the description of the *state* of a system for which the observable B has the value b' (this observable could, of course, as a special case, be A). In introducing this notion, we have explicitly recognized that an observable carries an intrinsic degeneracy. An analogous property exists for the complex number system in the usual formulation of the quantum theory; since it is commutative, it may be separated, as a phase, multiplicatively from the measurement symbols, along with the real numerical size of the coefficient (the local phases emerging at a later stage of development of the theory result in the $U(1)$ bundle characterizing electromagnetic interactions). In Eq.(2.1), the coefficient carries the same meaning implied by Schwinger¹; a system with A having the value a' has B with a distribution of values b' , represented by $\langle b'|a' \rangle$, to be eventually determined by the dynamical properties of the system. It carries, moreover, the additional information associated with the change of orientation of the number systems used for the A and B descriptions, in particular, relating the spectral values a' and b' *locally*. Such structures, with noncommutative degrees of freedom, associated with the manifold on which functions are to be defined (resulting, as we shall see, in sets of noncommutative functions) is sometimes referred to as noncommutative geometry^{7,8}.

The algebra of measurements is invariant under such automorphisms, which therefore constitute what might be considered a symmetry of the “apparatus”, in the sense of Piron⁹.

The generalized Pauli measurements of the second kind satisfy

$$M(a'; q; b')M(c'; p; d') = M(a'; q\langle b'|c' \rangle p; d'). \quad (2.3)$$

The “completeness sum”

$$\sum_{a'} M(a'; e_i; a') \equiv E_A(e_i) \quad (2.4)$$

satisfies an algebra isomorphic (it may be constructed to be star-isomorphic, as we show below) to \mathbf{H} , and it is called^{3,4} an element of the *left quaternion algebra* associated with the spectrum of A ($M(a'; e_0; a') \equiv M(a')$, so according to (1.2), $E_A(e_0) = I$).

With Eqs.(1.2) and (2.1), we see that, as for the complex theory,

$$\begin{aligned} \sum_{b'} M(a')M(b')M(a'') &= M(a')M(a'') = \delta(a', a'')M(a') \\ &= \sum_{b'} M(a'; \langle a'|b' \rangle \langle b'|a'' \rangle; a''), \end{aligned}$$

so that

$$\sum_{b'} \langle a'|b' \rangle \langle b'|a'' \rangle = \langle a'|a'' \rangle = \delta(a', a''). \quad (2.5)$$

More generally,

$$\begin{aligned} M(a')M(c') &= \sum_{b'} M(a')M(b')M(c') \\ &= \sum_{b'} M(a'; \langle a'|b' \rangle \langle b'|c' \rangle; c'), \end{aligned}$$

and hence

$$\sum_{b'} \langle a'|b' \rangle \langle b'|c' \rangle = \langle a'|c' \rangle. \quad (2.6)$$

As for the complex theory, the quantities $\langle a'|b' \rangle$ therefore act as transformation functions. To see their action on the change of description in the measurement symbols, consider the relation

$$M(a'; q; b') = \sum_{c'} M(a'; q; b') M(c') = \sum_{c'} M(a'; q \langle b'|c' \rangle; c'). \quad (2.7)$$

This transformation law can be written in a form more analogous to that of the complex theory by using elements of the left algebra:

$$\begin{aligned} \sum_{c'} M(a'; q; c') E_C(\langle b'|c' \rangle) &= \sum_{c', c''} M(a'; q; c') M(c''; \langle b'|c' \rangle; c'') \\ &= \sum_{c'} M(a'; q \langle b'|c' \rangle; c') \\ &= M(a'; q; b'), \end{aligned} \quad (2.8)$$

according to Eq. (2.7). Similarly, one finds

$$M(a'; q; b') = \sum_{c'} E_C(\langle c'|a' \rangle) M(c'; q; b'). \quad (2.9)$$

Following Schwinger¹, we define the conjugation † for which $M(a'; e_0; b') \equiv M(a', b')$ satisfies

$$M(a', b')^\dagger = M(b', a'), \quad (2.10)$$

and

$$(M(a', b') M(c', d'))^\dagger = M(d', c') M(b', a'). \quad (2.11)$$

If we supplement the definition (2.2) by defining the action of † on the number system entering the generalized Pauli measurement of the second kind through

$$M(a'; q; b')^\dagger = M(b'; q^*; a'), \quad (2.12)$$

where $*$ is the conjugation induced by † on the number system \mathbf{H} , it then follows from (2.11) that $\langle b'|c' \rangle^* = \langle c'|b' \rangle$. Eqs. (2.4) and (2.12) imply that $E_A(p)^\dagger = E_A(p^*)$. If we identify the action of this conjugation with an involutory automorphism isomorphic to that of \mathbf{H} , the left algebra is then star-isomorphic to \mathbf{H} . Eq.(2.6) then becomes, for $c' = a'$,

$$\sum_{b'} |\langle a'|b' \rangle|^2 = 1, \quad (2.13)$$

providing a formal probability interpretation, as for the complex quantum theory, for the absolute square of the amplitude $\langle a'|b' \rangle$. \natural

\natural The use of Clifford algebras of higher order than the C_2 quaternions requires some modification (e.g., introduction of the trace) to achieve an analogous result, since they are not normed division algebras. This more general case will be discussed elsewhere.

In the quaternionic theory, a real linear map corresponding to (1.7) from the measurement algebra to \mathbf{H} can be defined, but has properties that are somewhat more complicated.

Let us define a map from the measurement algebra to \mathbf{H} by

$$\text{tr}_A M(a'; q; b') = q \langle b' | a' \rangle \quad (2.14)$$

and

$$\text{tr}_B M(a'; q; b') = \langle b' | a' \rangle q. \quad (2.15)$$

These are not, in general, equal (in the complex case, they are equal, and independent of the basis used for the evaluation). It follows, however, from (2.7), that

$$\text{tr}_C M(a'; q; b') = \sum_{c'} \langle c' | a' \rangle q \langle b' | c' \rangle, \quad (2.16)$$

Taking the real part of (2.14) or (2.15), equivalent to carrying out an additional trace over the matrix representation of \mathbf{H} , removes this dependence; the resulting operation, called the *total trace*⁴, admits cycling the transformation coefficients. Adler^{4,10,11} has shown that this total trace operation plays an essential role in constructing a dynamical theory, and permits a wide generalization of the structure (in its application to quantum field theory, a graded total trace is used).

Elements of the measurement algebra called *operators* can be constructed in analogy with Schwinger's definition. A general operator is taken to have the form

$$\mathcal{O} = \sum_{a', b'} M(a'; \langle a' | \mathcal{O} | b' \rangle; b'). \quad (2.17)$$

In another basis,

$$\mathcal{O} = \sum_{a', c'} M(a'; \langle a' | \mathcal{O} | c' \rangle; c'). \quad (2.18)$$

Using (2.8) to rewrite (2.18), i.e.,

$$\mathcal{O} = \sum_{a', b', c'} M(a'; \langle a' | \mathcal{O} | c' \rangle \langle c' | b' \rangle; b'),$$

we see that

$$\langle a' | \mathcal{O} | b' \rangle = \sum_{c'} \langle a' | \mathcal{O} | c' \rangle \langle c' | b' \rangle; \quad (2.19)$$

the quantities $\{\langle a' | b' \rangle\}$ therefore act as transformation functions on the quaternionic coefficients characterizing the operators as well. That the representation (2.17) induces a homomorphic structure on these coefficients can be seen by examining the representation of a product of two operators \mathcal{O}_1 and \mathcal{O}_2 :

$$\begin{aligned} \mathcal{O}_1 \mathcal{O}_2 &= \sum_{a', b', a'', b''} M(a'; \langle a' | \mathcal{O}_1 | b' \rangle; b') M(a''; \langle a'' | \mathcal{O}_2 | b'' \rangle; b'') \\ &= \sum_{a', b', a'', b''} M(a'; \langle a' | \mathcal{O}_1 | b' \rangle \langle b' | a'' \rangle \langle a'' | \mathcal{O}_2 | b'' \rangle; b''); \end{aligned} \quad (2.20)$$

With the help of (2.19), one sees that (2.20) can be written as

$$\mathcal{O}_1 \mathcal{O}_2 = \sum_{a', b'} M(a'; \sum_{a''} \langle a' | \mathcal{O}_1 | a'' \rangle \langle a'' | \mathcal{O}_2 | b' \rangle; b'). \quad (2.21)$$

Returning to the general definition (2.17), the transformation theory we have described permits us to rewrite it entirely in terms of the spectrum of a single observable C , i.e.,

$$\mathcal{O} = \sum_{c', c''} M(c'; \langle c' | \mathcal{O} | c'' \rangle; c''); \quad (2.22)$$

if $\mathcal{O}^\dagger = \mathcal{O}$, i.e., a self-conjugate operator of the measurement algebra, it follows that

$$\langle c' | \mathcal{O} | c'' \rangle^* = \langle c'' | \mathcal{O} | c' \rangle. \quad (2.23)$$

It follows from the spectral theory of a quaternionic self-adjoint operator^{3,4}, realized in this case by the matrix $\{\langle c' | \mathcal{O} | c'' \rangle\}$, that there exists a unitary transformation to a representation in which the matrix is diagonal. Expressing the operators $M(c'; q; c'')$ in terms of the spectral family of another observable, say, A , one obtains

$$\mathcal{O} = \sum_{a', a'', c', c''} M(a'; \langle a' | c' \rangle \langle c' | \mathcal{O} | c'' \rangle \langle c'' | a'' \rangle; a''). \quad (2.24)$$

The transformation coefficients act on the representative of \mathcal{O} as a unitary transformation; with an appropriate choice of A , the resulting matrix becomes diagonal, and the operator then takes on the form

$$\mathcal{O} = \sum_{a'} \mathcal{O}_{a'} M(a'), \quad (2.25)$$

where $\{\mathcal{O}_{a'}\}$ are real. The trace,

$$\text{tr} \mathcal{O} = \sum_{a'} \mathcal{O}_{a'},$$

is then independent of the complete set used for the evaluation.

In the case of a general operator, (2.22) is clearly not, in general, diagonalizable to a real spectrum, and the trace is representation dependent. For example,

$$\text{tr}_A \mathcal{O} = \sum_{a', c', c''} \langle a' | c' \rangle \langle c' | \mathcal{O} | c'' \rangle \langle c'' | a' \rangle,$$

which is not equal to

$$\text{tr}_C \mathcal{O} = \sum_{c'} \langle c' | \mathcal{O} | c' \rangle;$$

for the *total* trace, the transformation coefficients can be cycled, and the result becomes independent of the basis.

Again, following Schwinger¹, a natural representation for the quaternionic measurement algebra is

$$M(a'; q; b') = |a'\rangle q \langle b'|, \quad (2.26)$$

with the rule

$$\langle a'| \cdot |b'\rangle = \langle a'|b'\rangle, \quad (2.27)$$

i.e., the formation of the scalar product. We then see that if

$$M(a', b')|c'\rangle = |a'\rangle \langle b'|c'\rangle,$$

it follows that

$$M(a'; q; b')|c'\rangle = |a'\rangle (q \langle b'|c'\rangle q^{-1}) q, \quad (2.28)$$

inducing an automorphism on the algebra of transformation coefficients.

This representation clearly reproduces all of the definitions and results stated above, and corresponds to the structure of a quaternionic Hilbert space.^{2,3,4}

3. Identical particles and quantum fields

We now turn to the formulation of the many-body problem and the construction of quantum fields, for the case that the transformation functions belong to \mathbf{H} , according to the algebraic method of Schwinger¹.

Again, we think of every element (system) in the beam, on which filtering is performed, as constructed of a number N of identical subsystems, each with an identical set of properties A, B, C, \dots . Enumerating the possible value these properties can acquire as a'_i, b'_i, \dots for $i = 1, 2, \dots, N$ labelling the subsystem, the subsystem filters are denoted as $M(a'_i), M(b'_i), \dots$. A symmetrical filter for a system of such subsystems can then be constructed as the (direct) sum

$$\sum_i M(a'_i) = \mathbf{M}(a'). \quad (3.1)$$

Recall that the interpretation of the sum of measurement symbols over possible values a' of A , as in Eq.(1.2), refers to the retention of elements of the beam in some storage facility. We may therefore consider the outcome of the action of (3.1) as the collection of N -body systems for which the i^{th} subsystem has the value a'_i of A .

I shall assume that the value of a variable occurring on one subsystem is independent of the values occurring on other subsystems, hence the measurement symbols $\{M(a'_i; q; b'_i)\}$ commute for $i \neq j$. The composite measurements

$$\mathbf{M}(a'; q; b') = \sum_i M(a'_i; q; b'_i)$$

should then form a closed algebra under commutation, as for the Abelian case. Some care must be taken, however, in the formation of numerical coefficients as the result of a product of two measurement symbols corresponding to the same subsystem. As pointed out in the one-particle case for the interpretation of $M(b'; q; a')$, after selection of subsystems with

value a'_i of A , the element of the beam (the whole system) is passed with the value b'_i of B , and an automorphism is generated on the numbers used to describe the state of the subsystem. As we shall see, the requirement that the system retain its symmetry after the measurement requires that this automorphism be induced by this operation on all of the subsystems in turn. The phase disturbance induced on one subsystem propagates, by means of the manifest symmetry of the system, to all of the subsystems in the sense of a derivation (corresponding to the additive effect of an infinitesimal transformation on a product, i.e., the Leibniz rule).

Consider, for example, the two body case, where we represent the physical state as (according to Bose-Einstein or Fermi-Dirac symmetry)

$$\Psi(c', d') = \frac{1}{\sqrt{2}}(|c'_1\rangle \otimes |d'_2\rangle \pm |d'_1\rangle \otimes |c'_2\rangle). \quad (3.1)$$

The indices refer to the particle names, and c' , d' specify the states of the subsystems 1, 2. The direct action of $\mathbf{M}(a', b')$ on $\Psi(c', d')$ is given (naively) according to the relations obtained in the one-particle theory, by

$$\begin{aligned} (M(a'_1, b'_1) + M(a'_2, b'_2)) \frac{1}{\sqrt{2}}(|c'_1\rangle \otimes |d'_2\rangle \pm |d'_1\rangle \otimes |c'_2\rangle) \\ = \frac{1}{\sqrt{2}} \{ |a'_1\rangle \langle b'|c'\rangle \otimes |d'_2\rangle \pm |a'_1\rangle \langle b'|d'\rangle \otimes |c'_2\rangle \\ + |c'_1\rangle \otimes |a'_2\rangle \langle b'|d'\rangle \pm |d'_1\rangle \otimes |a'_2\rangle \langle b'|c'\rangle \}, \end{aligned} \quad (3.2)$$

where we have assumed the equivalence $\langle b'_i|c'_i\rangle \equiv \langle b'|c'\rangle$ for all transformation coefficients. In the Abelian case, this result obviously retains its symmetry in the final state, but it does not in the non-Abelian quaternionic theory. Symmetrizing (3.2), one finds

$$\begin{aligned} \mathbf{M}(a', b')\Psi(c', d') = \frac{1}{2} \frac{1}{\sqrt{2}} \{ |a'_1\rangle \langle b'|c'\rangle \otimes |d'_2\rangle \pm |d'_1\rangle \langle b'|c'\rangle \otimes |a'_2\rangle \\ \pm |a'_1\rangle \langle b'|d'\rangle \otimes |c'_2\rangle + |c'_1\rangle \langle b'|d'\rangle \otimes |a'_2\rangle \\ + |c'_1\rangle \otimes |a'_2\rangle \langle b'|d'\rangle \pm |a'_1\rangle \otimes |c'_2\rangle \langle b'|d'\rangle \\ \pm |d'_1\rangle \otimes |a'_2\rangle \langle b'|c'\rangle + |a'_1\rangle \otimes |d'_2\rangle \langle b'|c'\rangle \}. \end{aligned} \quad (3.3)$$

Collecting terms, (3.3) becomes

$$\begin{aligned} \mathbf{M}(a', b')\Psi(c', d') = \frac{1}{\sqrt{2}} \{ |a'_1\rangle \otimes |d'_2\rangle \bullet \langle b'|c'\rangle \pm |d'_1\rangle \otimes |a'_2\rangle \bullet \langle b'|c'\rangle \\ \pm |a'_1\rangle \otimes |c'_2\rangle \bullet \langle b'|d'\rangle + |c'_1\rangle \otimes |a'_2\rangle \bullet \langle b'|d'\rangle \} \\ = \Psi(a', d') \bullet \langle b'|c'\rangle + \Psi(c', a') \bullet \langle b'|d'\rangle, \end{aligned} \quad (3.5)$$

where we have defined

$$\Psi(a', b') \bullet q = \frac{1}{2} \frac{1}{\sqrt{2}} \{ |a'\rangle q \otimes |b'\rangle + |a'\rangle \otimes |b'\rangle q \}, \quad (3.6)$$

and, similarly, with factor $1/N$ in place of the $1/2$ in (3.6), for any N . The \bullet -product arises from the requirement of symmetry of the state after the two-body measurement. We thus find the rule that the generation of a quaternion coefficient by the action of a filter must be additively distributed, as a derivation, among the factors of the tensor product in the representation of an N -body state.

We remark that the complete Fock space contains the vacuum as well, and hence (3.6) should be written as

$$\Psi(a', b') \bullet q = \frac{1}{3} \{ |a'\rangle q \otimes |b'\rangle \otimes |0\rangle + |a'\rangle \otimes |b'\rangle q \otimes |0\rangle + |a'\rangle \otimes |b'\rangle \otimes |0\rangle q \}. \quad (3.7)$$

We now investigate the possibility of closing the algebra of

$$\mathbf{M}(a'; q; b')$$

under commutation in a form analogous to that obtained by Schwinger for the Abelian case, i.e.,

$$[\mathbf{M}(a'; q; b'), \mathbf{M}(c'; p; d')] = \mathbf{M}(a'; q \langle b'|c'\rangle p; d') - \mathbf{M}(c', p \langle d'|a'\rangle q; b'), \quad (3.8)$$

where we have suppressed reference to the \bullet -product occurring in sequence at every multiplication.

I shall argue the validity of this result, in the limit $N \rightarrow \infty$, by studying here the structure of the two-body case, and indicating the form of the general case.

Let us apply $\mathbf{M}(e'; p; f')$ to the result (3.5). One obtains, in the same way,

$$\begin{aligned} \mathbf{M}(e'; p; f') \mathbf{M}(a'; q; b') \Psi(c', d') \\ &= \mathbf{M}(e'; p; f') \{ \Psi(a', d') \bullet \langle b'|c'\rangle \bullet q + \Psi(c', a') \bullet \langle b'|d'\rangle \bullet q \} \\ &= \Psi(e', d') \bullet \langle b'|c'\rangle \bullet q \bullet \langle f'|a'\rangle \bullet p \\ &\quad + \Psi(a', e') \bullet \langle b'|c'\rangle \bullet q \bullet \langle f'|d'\rangle \bullet p \\ &\quad + \Psi(e', a') \bullet \langle b'|d'\rangle \bullet q \bullet \langle f'|c'\rangle \bullet p \\ &\quad + \Psi(c', e') \bullet \langle b'|d'\rangle \bullet q \bullet \langle f'|a'\rangle \bullet p \}. \end{aligned} \quad (3.9)$$

In reverse order,

$$\begin{aligned} \mathbf{M}(a'; q; b') \mathbf{M}(e'; p; f') \Psi(c', d') \\ &= \mathbf{M}(a'; q; b') \{ \Psi(e', d') \bullet \langle f'|c'\rangle \bullet p + \Psi(c', e') \bullet \langle f'|d'\rangle \bullet p \} \\ &= \Psi(a', d') \bullet \langle f'|c'\rangle \bullet p \bullet \langle b'|e'\rangle \bullet q \\ &\quad + \Psi(e', a') \bullet \langle f'|c'\rangle \bullet p \bullet \langle b'|d'\rangle \bullet q \\ &\quad + \Psi(a', e') \bullet \langle f'|d'\rangle \bullet p \bullet \langle b'|c'\rangle \bullet q \\ &\quad + \Psi(c', a') \bullet \langle f'|d'\rangle \bullet p \bullet \langle b'|e'\rangle \bullet q \}. \end{aligned} \quad (3.10)$$

The difference between these, the commutator, contains the terms

$$\begin{aligned} &\Psi(a', e') \bullet \{ \langle b'|c'\rangle \bullet q \bullet \langle f'|d'\rangle \bullet p - \langle f'|d'\rangle \bullet p \bullet \langle b'|c'\rangle \bullet q \} \\ &\quad + \Psi(e', a') \bullet \{ \langle b'|d'\rangle \bullet q \bullet \langle f'|c'\rangle \bullet p - \langle f'|c'\rangle \bullet p \bullet \langle b'|d'\rangle \bullet q \}. \end{aligned} \quad (3.11)$$

Each of these contributions cancels in the Abelian case. Due to the lack of commutativity, cancellation does not, in general, take place in the quaternion case. However, if we study the effect of the commutator of the generalized measurement symbols on a many body state ($N \geq 4$), cancellations take place among all terms which contain each factor linearly (the \bullet -product is linearly distributive), since these factors will occur symmetrically at every position. The number of these terms is (we include the vacuum factor in the counting)

$$\frac{(N+1)!}{(N-3)!4!} \sim N^4$$

as $N \rightarrow \infty$. The remaining terms contain two, three, or four (non-commuting) factors at every occupied site, with frequency of occurrence asymptotically N^2 , N^3 and N , respectively; although these terms do not cancel, the four \bullet -products induce a factor N^4 in the denominator, and hence all other combinations other than the linearly distributed one vanish in the limit (the linearly distributed terms cancel exactly). The \bullet -product of quaternion coefficients is therefore *Abelian* in the limit $N \rightarrow \infty$.

The remaining terms, for which no cancellations occur, close the algebra in a form precisely analogous to the Abelian case, (1.13). The corresponding terms in the two-body case, reading from (3.9) and (3.10), are

$$\begin{aligned} & \Psi(e', d') \bullet \langle b' | c' \rangle \bullet q \bullet \langle f' | a' \rangle \bullet p \\ & + \Psi(c', e') \bullet \langle b' | d' \rangle \bullet q \bullet \langle f' | a' \rangle \bullet p \\ & - \Psi(a', d') \bullet \langle f' | c' \rangle \bullet p \bullet \langle b' | e' \rangle \bullet q \\ & - \Psi(c', a') \bullet \langle f' | d' \rangle \bullet p \bullet \langle b' | e' \rangle \bullet q. \end{aligned} \tag{3.12}$$

In the general case of N particles, there are N^4 non-cancelling contributions to each of these terms, so dividing by N^4 gives a mean value for selection of subsystems contained, e.g., in $\Psi(d'_1, d'_2, \dots, d'_\ell, \dots, d'_N)$ projected to f' or b' . There remain infinite sums, for $N \rightarrow \infty$, on $i = 1, 2 \dots N$ containing factors $\langle b' | d'_\ell \rangle, \langle f' | d'_\ell \rangle$, and we assume sufficiently rapid convergence (in the sense of Fourier transform) as $\ell \rightarrow \infty$.

Using the rule of successively applying the \bullet -products to the left, one easily sees that (3.12) arises from the application of

$$\mathbf{M}(e'; p \langle f' | a' \rangle q; b') - \mathbf{M}(a'; q \langle b' | e' \rangle p; f')$$

to the Fock space vector $\Psi(c', d')$, where we understand that the quaternionic factors appearing in the arguments must be applied as \bullet -products. The same conclusion follows in the general case.

The Schwinger many-body algebra therefore closes in a form precisely analogous to that of the Abelian case (1.13),

$$\begin{aligned} & [\mathbf{M}(e'; p; f') \mathbf{M}(a'; q; b') - \mathbf{M}(a'; q; b') \mathbf{M}(e'; p; f')] \\ & = \mathbf{M}(e'; p \langle f' | a' \rangle q; b') - \mathbf{M}(a'; q \langle b' | e' \rangle p; f'). \end{aligned} \tag{3.13}$$

4. Factorization in terms of quantum fields

In this section, I demonstrate that the annihilation- creation operators obtained by Razon and Horwitz⁵, in the limit as the occupation number $N \rightarrow \infty$, provide an effective factorization of the quaternionic many-body measurement operators. I review first the definition of these operators, and show that the operators $\psi^\dagger(a') \bullet q$, $\psi(a') \bullet q$, where $\psi^\dagger(a')$, $\psi(a')$ are Bose-Einstein or Fermi-Dirac creation and annihilation operators on the infinite occupation number Fock space, are well-defined. It then follows that

$$\mathbf{M}(a'; q; b') = \psi(a')^\dagger \bullet q \psi(b') \quad (4.1)$$

provides a representation of the Schwinger algebra (3.11).

Razon and Horwitz⁵ define creation operators on the N -particle Fock space with Bose-Einstein or Fermi-Dirac symmetry with the property

$$\psi_N^\dagger(g) \Psi(g_1, \dots, g_N) = \Psi(g_1, \dots, g_N, g), \quad (4.2)$$

where I make explicit the notation for the (quaternionic) wave functions for the states of the constituents (the eigenfunction $g^{a'}$ corresponds to $|a'\rangle$ in the notation used above). A scalar product between such Ψ 's is given in ref. 5 as a map of pairs Ψ_1, Ψ_2 into \mathbf{H} with positive definite norm (the symplectic tensor product and scalar products worked out earlier by Biedenharn and Horwitz³ constitute a special case, but is not a completely quaternion covariant structure). By means of the Riesz theorem, the corresponding annihilation operators were found to have the property

$$\begin{aligned} \psi_N(f) \Psi(g_1, \dots, g_N) \\ = \sum_{j=1}^N (\pm)^{N-j} \Psi(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \bullet \langle f | g_j \rangle_N, \end{aligned} \quad (4.3)$$

where

$$\langle f | g_j \rangle_N = \frac{1}{N+2} [N \langle f | g_j \rangle + 2 \text{Re} \langle f | g_j \rangle], \quad (4.4)$$

and $\langle f | g \rangle$ is the quaternionic (one-particle) scalar product^{3,4}. We see from (4.4) the explicit dependence of these operators on the occupation number of the states. The commutation (anticommutation) relations of these operators is given by

$$\begin{aligned} [\psi_{N+1}(f) \psi_N^\dagger(g) \mp \frac{N+2}{N+3} \psi_{N-1}(g) \psi_N(f)] \Psi(g_1, \dots, g_N) \\ = \Psi(g_1, \dots, g_N) \bullet \langle f | g \rangle_{N+1}, \\ \pm \frac{1}{N+3} \sum_{j=1}^N (\pm)^{N-j} \Psi(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N \langle f | g_j \rangle), \end{aligned} \quad (4.5)$$

which are deformed⁶ for finite N .

It was shown that if the tail of the sequence $\{g_j\}$, $j = 1, \dots, N$, approaches orthonormality for large j , then the norm of the last term in (4.5) approaches zero as $N \rightarrow \infty$. It is also true that

$$\langle f | g \rangle_N \rightarrow \langle f | g \rangle$$

for $N \rightarrow \infty$, and hence the relations (4.5) become, in this limit,

$$[\psi(f)\psi^\dagger(g) \mp \psi^\dagger(g)\psi(f)]\Psi(g_1, \dots) = \Psi(g_1, \dots) \bullet \langle f|g \rangle. \quad (4.6)$$

The quantum fields, in this limit, become independent of N , and the commutation (anticommutation) relations are no longer deformed. The \bullet -product on the right side of (4.6) corresponds to a limiting process, and is well-defined (through scalar products) by the same argument for which the second term on the right hand side of (4.5) vanishes.

Since the state Ψ includes the vacuum as a factor (denoted by $\otimes 1$ in the algebraic structure of the tensor product studied in ref. 5),

$$\Psi(g_1, \dots, g_N) \bullet q = \frac{1}{N+1} \left\{ \sum_{j=1}^N \Psi(g_1, \dots, g_{j-1}, g_j q, g_{j+1}, \dots, g_N) + q \Psi(g_1, \dots, g_N) \right\}, \quad (4.7)$$

where the last terms corresponds to multiplication of the vacuum factor (on the right, if there has been a previous quaternion factor) by q .⁵ Creating another particle on this state by means of $\psi^\dagger(f)$, we have

$$\begin{aligned} \psi_N^\dagger(f) (\Psi(g_1, \dots, g_N) \bullet q) &= \frac{1}{N+1} \left\{ \sum_{j=1}^N \Psi(g_1, \dots, g_{j-1}, g_j q, g_{j+1}, \dots, g_N, f) \right. \\ &\quad \left. + q \Psi(g_1, \dots, g_N, f) \right\}. \end{aligned} \quad (4.8)$$

On the other hand,

$$\begin{aligned} (\psi_N^\dagger(f) \Psi(g_1, \dots, g_N)) \bullet q &= \Psi(g_1, \dots, g_N, f) \bullet q \\ &= \frac{1}{N+2} \left\{ \sum_{j=1}^N \Psi(g_1, \dots, g_{j-1}, g_j q, g_{j+1}, \dots, g_N, f) \right. \\ &\quad \left. + \Psi(g_1, \dots, g_N, f q) + q \Psi(g_1, \dots, g_N, f) \right\}. \end{aligned} \quad (4.9)$$

The difference between (4.8) and (4.9) is given by

$$\begin{aligned} (N+2) (\psi_N^\dagger(f) \Psi(g_1, \dots, g_N)) \bullet q \\ - (N+1) \psi_N^\dagger(f) (\Psi(g_1, \dots, g_N) \bullet q) \\ = \psi_N^\dagger(f q) \Psi(g_1, \dots, g_N). \end{aligned} \quad (4.10)$$

The right hand side is bounded, as is every term resulting from the \bullet -products, so that for $N \rightarrow \infty$,

$$(\psi^\dagger(f) \Psi(g_1, \dots)) \bullet q = \psi^\dagger(f) (\Psi(g_1, \dots) \bullet q). \quad (4.11)$$

It then follows from (4.10), in this limit, that the field operators have the formal quaternionic linearity property

$$\psi^\dagger(f q) \Psi(g_1, \dots) = \psi^\dagger(g) \Psi(g_1, \dots) \bullet q. \quad (4.12)$$

The action of $\psi^\dagger(f)$ is, according to (4.11), of *linear type*³ with respect to the \bullet -product with a quaternion number, and the operator

$$\psi^\dagger(f) \bullet q \Psi \equiv \psi^\dagger(f) \Psi \bullet q$$

is therefore well defined.

In a similar way, we may show that

$$\psi(f)(\Psi(g_1, \dots) \bullet q) = (\psi(f)\Psi(g_1, \dots)) \bullet q, \quad (4.13)$$

and hence the operator $\psi(f) \bullet q$ is also well-defined in the limiting case of infinite N . To see this, note that typical terms occurring on the right hand side of (4.13), for finite N , are

$$\begin{aligned} & (\psi_N(f)\Psi(g_1, \dots, g_N)) \bullet q \\ &= \sum_{j=1}^N (\pm)^{N-j} \Psi(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \bullet \langle f|g_j \rangle \bullet q \\ &= \frac{1}{N^2} \sum_{j=1}^N (\pm)^{N-j} \{ \Psi(g_1 \langle f|g_j \rangle, g_2, \dots, g_k q, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \\ &+ \Psi(g_1, \dots, g_k \langle f|g_j \rangle q, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \\ &+ \langle f|g_j \rangle q \Psi(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \}. \end{aligned} \quad (4.14)$$

There are

$$\frac{(N-1)!}{2!(N-3)!} \sim N^2$$

terms of the first type, N of the second and one of the third, for each j . Hence only terms

of the first type contribute for $N \rightarrow \infty$. For the left side of (4.13), one obtains

$$\begin{aligned}
& \psi_N(f) (\Psi(g_1, \dots, g_N) \bullet q) \\
&= \frac{1}{N+1} \left\{ \sum_{\substack{j=1 \\ k \neq j}}^N (\pm)^{N-j} \Psi(g_1, \dots, g_k q, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \bullet \langle f|g_j \rangle_N \right. \\
&+ \sum_{j=1}^N (\pm)^{N-j} \Psi(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \bullet \langle f|g_j q \rangle_N \\
&+ q \Psi(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \bullet \langle f|g_j \rangle_N \\
&= \frac{1}{N(N+1)} \sum_{\substack{j=1 \\ k \neq j}}^N (\pm)^{N-j} \{ \Psi(g_1 \langle f|g_j \rangle, \dots, g_k q, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \\
&+ \dots + \Psi(g_1, \dots, g_k q \langle f|g_j \rangle, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \\
&+ q \Psi(g_1, \dots, g_k \langle f|g_j \rangle, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \\
&+ \Psi(g_1, \dots, g_k \langle f|g_j q \rangle, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \\
&+ \langle f|g_j \rangle \Psi(g_1, \dots, g_k q, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \\
&+ \frac{1}{N(N+1)} \sum_{j=1}^N (\pm)^{N-j} \{ \langle f|g_j q \rangle \Psi(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \\
&+ q \langle f|g_j \rangle \Psi(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N) \}.
\end{aligned} \tag{4.15}$$

Terms of the first type, from distributing q and the annihilation factor on separate constituents, are $O(N^2)$, for each j , in number; terms of the second type, where the distribution of the annihilation factor coincides with terms with factor q , are $O(N)$ for each j , as are terms of the third type, from the distribution of q on the vacuum contribution from the annihilation. Terms of the fourth type, from the annihilation of $g_j q$ factors are also $O(N)$. The last two, the vacuum contribution of the annihilation of $g_j q$ terms, and the product of the two vacuum factors from $\bullet q$ and the annihilation of g_j , are each just a single term. In the limit $N \rightarrow \infty$, only terms of the first type survive, and these are identical to those remaining in (4.14).

We now assert that

$$\mathbf{M}(f; q; g) = \psi^\dagger(f) \bullet q \psi(g)$$

provides an effective factorization of the many-body measurement symbols, on states for which $N \rightarrow \infty$. Consider

$$\begin{aligned}
& \mathbf{M}(f'; p; g') \mathbf{M}(f; q; g) \Psi(g_1, \dots) \\
&= \psi^\dagger(f') \bullet p \psi(g') \psi^\dagger(f) \bullet q \psi(g) \Psi(g_1, \dots) \\
&= \psi^\dagger \bullet p \psi(g') \psi^\dagger(f) \psi(g) \Psi(g_1, \dots) \bullet q.
\end{aligned} \tag{4.16}$$

Using the identity

$$\psi(g') \psi^\dagger(f) = \pm \psi^\dagger(f) \psi(g') + [\psi(g'), \psi^\dagger(f)]_\mp,$$

so that (4.15) becomes, with (4.6),

$$\begin{aligned}
& \mathbf{M}(f'; p; g') \mathbf{M}(f; q; g) \Psi(g_1, \dots) \\
&= (\pm) \psi^\dagger(f') \bullet p \psi^\dagger(f) \psi(g') (\psi(g) \Psi(g_1, \dots)) \bullet q \\
&+ \psi^\dagger(f') \bullet p (\psi(g) \Psi(g_1, \dots)) \bullet q \bullet \langle g' | f \rangle \\
&= (\pm) \psi^\dagger(f') \psi^\dagger(f) (\psi(g') (\psi(g) \Psi(g_1, \dots)) \bullet q) \bullet p \\
&+ \psi^\dagger(f') \bullet p \bullet \langle g' | f \rangle \bullet q \psi(g) \Psi(g_1, \dots).
\end{aligned} \tag{4.17}$$

In reverse order,

$$\begin{aligned}
& \mathbf{M}(f; q; g) \mathbf{M}(f'; p; g') \Psi(g_1, \dots) \\
&= (\pm) \psi^\dagger(f) \psi^\dagger(f') (\psi(g) (\psi(g') \Psi(g_1, \dots)) \bullet p) \bullet q \\
&+ \psi^\dagger(f) \bullet q \bullet \langle g | f' \rangle \bullet p \psi(g') \Psi(g_1, \dots).
\end{aligned} \tag{4.18}$$

The relations (4.11) and (4.13) permit us to eliminate the association parentheses in the first terms of (4.17) and (4.18), and the difference is then

$$\begin{aligned}
& \mathbf{M}(f'; p; g') \mathbf{M}(f; q; g) - \mathbf{M}(f; q; g) \mathbf{M}(f'; p; g') \Psi(g_1, \dots) \\
&= (\pm) \psi^\dagger(f') \psi^\dagger(f) \psi(g') \psi(g) \Psi(g_1, \dots) \bullet (q \bullet p - p \bullet q) \\
&+ \{ \psi^\dagger(f') \bullet p \bullet \langle g' | f \rangle \bullet q \psi(g) \\
&- \psi^\dagger(f) \bullet q \bullet \langle g | f' \rangle \bullet p \psi(g') \} \Psi(g_1, \dots).
\end{aligned} \tag{4.19}$$

The first term of (4.19), in the distributive product commutator, contains, as in (3.11), cancellations of the $O(1)$ terms, and hence vanishes in the $N \rightarrow \infty$ limit. The second term of (4.19) reproduces (3.12).

5. Discussion

The Schwinger algebra, based on fundamental notions of measurements and the kinematical independence of elements of the quantum ensemble, can be realized in terms of quaternionic quantum theory as well as in the framework of the usual complex Hilbert space. Extending the algebra to the many-body theory, Schwinger found that the Bose-Einstein and Fermi-Dirac quantum fields of the complex theory provide a representation of the commutation relations of the many-body algebra independently of the class of states of the theory. In the quaternionic case, I find that the many-body algebra cannot be closed consistently in the finite particle sector of the Fock space. If all physical states of the theory contain an infinite number of elementary excitations, i.e., in the infinite particle sector of the Fock space (the Haag theorem¹² asserts that there is no unitary transformation that connects the finite and infinite sectors), the algebra closes in Schwinger's form. In this sector, the distributive \bullet -product of quaternion coefficients constitutes an associative, Abelian algebra. A quaternion factor q occurring as a right multiplier of any constituent state in the tensor product may be extracted as a \bullet -product to the right of the Fock space

vector[(4.12)]; since such multipliers are Abelian (in the $N \rightarrow \infty$ limit), the result of extracting several such factors is independent of the order or their source. The \bullet -product of quaternion factors (in the infinite sector) is not homomorphic to the quaternion algebra \mathbf{H} , or to any finite algebra; it does not close.

The annihilation-creation operators of the quaternionic Fock space defined by Razon and Horwitz⁵ provide an effective representation for the factorization of this algebra in the limit $N \rightarrow \infty$. It is only in this limit that the commutation (anticommutation) relations of these operators are not deformed, and coincide formally with the commutation (anticommutation) relations of complex quantum field theory. The numerical coefficient arising from the commutation (anticommutation) of operators associated with different one particle states appears as a \bullet -product on vectors of the Fock space as well.

I have demonstrated that in the infinite sector, the distributed right multipliers may be considered as part of the action of the field operators themselves, which then form, in this way, an algebra with many of the properties of the usual complex field. Since the commutation (anticommutation) relations of the fields are of the same structure as in the usual complex theory, the treatment of dynamical problems, such as the construction of the S -matrix by diagrammatic techniques becomes accessible (with some suitable definition of the ground state, as in condensed matter theory).

In the case of finite N , the commutation (anticommutation) relations are deformed, and the Schwinger algebra is not satisfied for the many-body theory. The physical meaning of this sub-asymptotic regime, and the relation of the structure discussed here to the work of Adler and Millard^{11,13}, remain to be investigated.

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